Minimum density of identifying codes of king grids

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Abstract

A set $C \subseteq V(G)$ is an *identifying code* in a graph G if for all $v \in V(G)$, $C[v] \neq \emptyset$, and for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$, where $C[v] = N[v] \cap C$ and $N[v]$ denotes the closed neighbourhood of v in G. The minimum density of an identifying code in G is denoted by $d^*(G)$. In this paper, we study the density of king grids which are strong products of two paths. We show that for every king grid G, $d^*(G) \geq 2/9 = 0.222$. In addition, we show that this bound is attained only for king grids which are strong products of two infinite paths. Given a positive integer k, we denote by $\mathcal{K}_k$ the (infinite) king strip with k rows. We prove that $d^*(\mathcal{K}_3) = 1/3 = 0.333$, $d^*(\mathcal{K}_4) = 5/16 = 0.3125$, $d^*(\mathcal{K}_5) = 0.2666$ and $d^*(\mathcal{K}_6) = 5/18 = 0.2777$. We also prove that $\frac{2}{9} + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \frac{2}{9} + \frac{4}{9k}$ for every $k \geq 7$.

Keywords. identifying codes, king grids, Discharging Method.

1 Introduction

Given an integer $k \geq 1$, let $[k] = \{1, \dots, k\}$. Let G be a graph and v a vertex of G. The *neighbourhood* of v, denoted by $N(v)$, is the set of vertices adjacent to v in G, and the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$.

Given a set $C \subseteq V(G)$, the *identifier* of a vertex $v \in V(G)$ is $C[v] = N[v] \cap C$. We say that C is an *identifying code* of G if every vertex has non-empty identifier and, for all distinct u and v, u and v have distinct identifiers. Formally, C is an *identifying code* if

(i) for all $v \in V(G)$, $C[v] \neq \emptyset$, and

(ii) for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$.

It is easy and well-known that a graph G has an identifying code if and only if it contains no *twins* (vertices $u, v \in V(G)$ with $N[u] = N[v]$).

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the *ball of radius r in G* centered at v, that is $B_r(v) = \{x : \mathrm{dist}(v,x) \le r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C,G)$, is defined by

$$d(C,G) = \limsup_{r\to+\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} \ ,$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code of G.

The problem of finding low-density identifying codes was introduced in [14] in relation to fault diagnosis in arrays of processors. Here the vertices of an identifying code correspond to controlling processors able to check themselves and their neighbours. Thus the identifying property guarantees location of a faulty processor from the set of "complaining" controllers. Identifying codes are also applied for compact routing in networks [15, 16], emergency sensor networks in facilities [17], to model a location detection problem with sensor networks [18], or the analysis of secondary RNA structures [12].

Particular interest was dedicated to grids as many processor networks have a grid topology. There are three regular infinite planar grids, namely the hexagonal grid, the square grid and the triangular grid.

Regarding the infinite square grid $\mathcal{G}_S$, Cohen et al. [5] gave a periodic identifying code of $\mathcal{G}_S$ with density $7/20 = 0.35$. This density was later proved to be optimal by Ben-Haim and Litsyn [2]. Some papers also obtained results for square grids with finite number of rows. For any positive integer k, let $\mathcal{S}_k$ denote the square grid with k rows. Daniel, Gravier, and Moncel [10] showed that $d^*(\mathcal{S}_1) = 1/2 = 0.5$ and $d^*(\mathcal{S}_2) = 3/7 = 0.429$. They also showed that, for every $k \ge 3$, $\frac{7}{20} - \frac{1}{2k} \le d^*(\mathcal{S}_k) \le \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{2}{k}\right\}$. These bounds were recently improved by Bouznif et al. [3] who established

$$\frac{7}{20} + \frac{1}{20k} \le d^*(\mathcal{S}_k) \le \min\left\{\frac{2}{5}, \frac{7}{20} + \frac{3}{10k}\right\}.$$

They also proved $d^*(\mathcal{S}_3) = 3/7 = 0.429$. Recently, Jiang [13] proved $d^*(\mathcal{S}_4) = 11/28 = 0.393$ and $d^*(\mathcal{S}_5) = 0.38$.

Regarding the infinite triangular grid $\mathcal{G}_T$, Karpovsky et al. [14] showed that $d^*(\mathcal{G}_T) = 1/4 = 0.25$. Let $\mathcal{T}_k$ denote the triangular grid with k rows. In 2016, Dantas et al. [11] proved that $d^*(\mathcal{T}_1) = d^*(\mathcal{T}_2) = 1/2 = 0.5$, $d^*(\mathcal{T}_3) = d^*(\mathcal{T}_4) = 1/3 = 0.333$, $d^*(\mathcal{T}_5) = 0.3$, $d^*(\mathcal{T}_6) = 1/3 = 0.333$ and $d^*(\mathcal{T}_k) = 1/4 + 1/(4k)$ for every $k \ge 7$ odd. Moreover, they proved that $1/4 + 1/(4k) \le d^*(\mathcal{T}_k) \le 1/4 + 1/(2k)$ for every $k \ge 8$ even, and conjectured that $d^*(\mathcal{T}_k) = 1/4 + 1/(2k)$ for every $k \ge 8$ even.

Regarding the infinite hexagonal grid $\mathcal{G}_H$, the best known upper bound on $d^*(\mathcal{G}_H)$ is $3/7 = 0.429$ and comes from two identifying codes constructed by Cohen et al. [6]; these authors also proved a lower bound of $16/39 = 0.4102$. This lower bound was improved to $12/29 = 0.4138$ by Cranston and Yu [8]. Cukierman and Yu [9] further improved it to $5/12 = 0.4166$.

In this paper, we study *king grids*, which are strong products of two paths. Given two graphs G and H, the *strong product* of G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edge set :

$$\begin{aligned} E(G \boxtimes H) = & \quad \{(a,b)(a,b') \ : \ a \in V(G) \text{ and } bb' \in E(H)\} \\ & \cup \ \{(a,b)(a',b) \ : \ aa' \in E(G) \text{ and } b \in V(H)\} \\ & \cup \ \{(a,b)(a',b') \ : \ aa' \in E(G) \text{ and } bb' \in E(H)\}. \end{aligned}$$

The *two-way infinite path*, denoted by $P_{\mathbb{Z}}$, is the graph with vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} \ : \ i \in \mathbb{Z}\}$, and the *one-way infinite path*, denoted by $P_{\mathbb{N}}$, is the graph with vertex set $\mathbb{N}$ and edge set $\{\{i, i+1\} \ : \ i \in \mathbb{N}\}$. A *path* is a connected subgraph of $P_{\mathbb{Z}}$. In particular, for every positive integer k, the *finite path of length* $k-1$, denoted by P_k, is the subgraph of $P_{\mathbb{Z}}$ induced by $\{1, 2, \ldots, k\}$.

A *king grid* is the strong product of two (finite or infinite) paths. The *plane king grid* is $\mathcal{G}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{Z}}$, the *half-plane king grid* is $\mathcal{H}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{N}}$, the *quater-plane king grid* is $\mathcal{Q}_K = P_{\mathbb{N}} \boxtimes P_{\mathbb{N}}$, and the *king strip of height* k is $\mathcal{K}_k = P_{\mathbb{Z}} \boxtimes P_k$. Note that every king grid is an induced subgraph of $\mathcal{G}_K$.

In 2002, Charon et al. [4] proved that $d^*(\mathcal{G}_K)$ is $2/9 = 0.222$. They provided the tile depicted in Figure 1, which generates a periodic tiling of the plane with periods $(0, 6)$ and $(6, 0)$, yielding an identifying code C_∞ of the infinite king grid with density $2/9$.

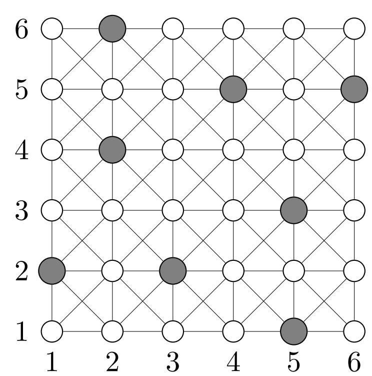

Figure 1: Tile generating an optimal identifying code of the infinite king grid. Black vertices are those of the code.

In 2013, Auger and Honkala investigated properties of watching systems in king grids, which is an extension of the notion of identifying code [1].

In this paper, we first prove that $\mathcal{G}_K$ is one king grid with the smallest identifying code density: for every king grid G, $d^*(G) \geq 2/9$ (Theorem 1). Our proof uses the Discharging Method. See Section 3 of [11] for a detailed presentation of this technique for identifying codes. The advantage of this method is that it allows us to give better lower bounds for some king grids. We first prove that $d^*(G) > 2/9$ for all finite king grids (Theorem 2). Then we consider king strips. In Section 3, we consider king strips of height at least 7. We prove that $d^*(\mathcal{K}_k) \geq 2/9 + \frac{8}{81k}$ for every $k \geq 7$ (Theorem 3). Moreover, we prove that $d^*(\mathcal{K}_k) \leq 2/9 + 6/(18k)$ if $k = 3p$, $d^*(\mathcal{K}_k) \leq 2/9 + 8/(18k)$ if $k = 3p+1$, and $d^*(\mathcal{K}_k) \leq 2/9 + 7/(18k)$ if $k = 3p+2$, for $p \in \mathbb{N}$ (Theorem 4). Clearly $d^*(\mathcal{K}_1) = 1/2$ (as $\mathcal{K}_1 = \mathcal{S}_1 = P_{\mathbb{Z}}$) and $\mathcal{K}_2$ has no identifying code, since $N[(a,1)] = N[(a,2)]$ for every $a \in \mathbb{Z}$. In Section 4, we show optimal identifying codes of king strips of height 3, 4, 5 and 6. We prove that $d^*(\mathcal{K}_3) = 1/3 = 0.333$, $d^*(\mathcal{K}_4) = 5/16 = 0.3125$, $d^*(\mathcal{K}_5) = 4/15 = 0.2666$ and $d^*(\mathcal{K}_6) = 5/18 = 0.2777$ (Theorems

6 to 9). All these results imply that $\mathcal{G}_K$, $\mathcal{H}_K$ and $\mathcal{Q}_K$ are the unique king grids having an identifying code with density $2/9 = 0.222$ (one can easily derive from C_∞ identifying codes with density $2/9$ of $\mathcal{H}_K$ and $\mathcal{Q}_K$).

2 General lower bound for king grids

Theorem 1. *If G is a (finite or infinite) king grid, then $d^*(G) \geq 2/9 = 0.222$.*

Proof. Let G be a king grid and C an identifying code of G. We set $U = V(G) \setminus C$. We shall prove that $d(C, G) \geq 2/9$.

We use the Discharging Method. The initial charge of a vertex v is 1 if $v \in C$ and 0 otherwise. We then apply some local discharging rules. We shall prove that the final charge of every vertex in G is at least $2/9$. This would imply the result.

Given a subset X of vertices in G and an integer $1 \leq i \leq 9$, we denote by X_i (resp. $X_{\geq i}$, $X_{\leq i}$) the set of vertices in X having exactly i vertices (resp. at least i vertices, at most i vertices) in their identifier defined by C. In particular, U_9 is empty. An *X-vertex* is a vertex in X and an *X-neighbour* is a neighbour in X.

A vertex is *full* if its eight neighbours in $\mathcal{G}_K$ are in G. Otherwise it is a *side vertex*. Observe that every side vertex has at most five neighbours in G.

Claim 1.1. *Two C_2-vertices are not adjacent.*

Proof. Suppose for a contradiction that two C_2-vertices u and v are adjacent. Then $C[u] = C[v] = \{u, v\}$, a contradiction. $\Diamond$

Claim 1.2. *If $(a,b) \in C$, $\{(a+1,b-1),(a+1,b),(a+1,b+1)\} \subseteq U$ and $(a+1,b) \notin U_1$, then at least one vertex of $\{(a+1,b-1),(a+1,b),(a+1,b+1)\}$ is in $U_{\geq 3}$.*

Proof. Suppose $(a,b) \in C$ and $\{(a+1,b-1),(a+1,b),(a+1,b+1)\} \subseteq U$. If $(a+1,b) \in U_2$, then its identifier is contained in either the identifier of $(a+1,b-1)$ or the one of $(a+1,b+1)$. Hence, one of these two vertices is in $U_{\geq 3}$. $\Diamond$

Claim 1.3. *.*

(i) Every C-vertex has at most one neighbour in U_1.

(ii) Every full C_2-vertex has at least three neighbours in $U_{\geq 3}$.

(iii) Every full C_3-vertex has a neighbour in $U_{\geq 3}$.

Proof. (i) All neighbours of a C-vertex v have v in their identifier. Since all identifiers are distinct, at most one of them is $\{v\}$.

(ii) Let $v = (a,b)$ be a full C_2-vertex and let w be its C-neighbour. Then v and w have two common neighbours x and y whose identifiers contain $\{v, w\}$. Since $\{v, w\}$ is the identifier of v, it cannot be the one of x or y which, consequently, must be in $U_{\geq 3}$. Furthermore, by symmetry, we may assume that $\{(a+1,b-1),(a+1,b),(a+1,b+1)\} \cap \{w,x,y\} = \emptyset$ and $(a+1,b) \notin U_1$.

Thus, by Claim 1.2, there is a vertex in $U_{\geq 3}$ in $\{(a+1,b-1),(a+1,b),(a+1,b+1)\}$ which is distinct from x and y.

(iii) Let $v=(a,b)$ be a full C_3-vertex and let u_1 and u_2 be its two neighbours in C. If a U-neighbour w of v is adjacent to both u_1 and u_2, then w is in $U_{\geq 3}$ and we have the result. If not, then u_1 and u_2 must be diagonal symmetric with respect to v, i.e. either $\{u_1,u_2\}=\{(a-1,b-1),(a+1,b+1)\}$ or $\{u_1,u_2\}=\{(a-1,b+1),(a+1,b-1)\}$. By symmetry, we may assume that we are in the first case. Now $\{u_1,v\}$ is in the identifiers of both $(a,b-1)$ and $(a-1,b)$, hence one of those must be in $U_{\geq 3}$. $\Diamond$

Claim 1.4. *Every C_1-vertex (a,b) has no neighbour in U_1 and at most six neighbours in U_2. Furthermore, if it has six neighbours in U_2, then either $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\}\subseteq C$ or $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\}\subseteq C$.*

Proof. Let $v=(a,b)$ be a C_1-vertex. Its identifier is $\{v\}$. Moreover, all its neighbours have an identifier containing v but distinct from $\{v\}$. Hence, they are not in U_1.

Let us no prove that v has at most six neighours in U_2. If v is a side vertex, then it is trivially true, so we may assume that v is a full vertex.

By Claim 1.2 and symmetry, there is a vertex of $U_{\geq 3}$ in each of the sets $\{(a+1,b-1),(a+1,b),(a+1,b+1)\}$, $\{(a-1,b-1),(a-1,b),(a-1,b+1)\}$, $\{(a-1,b-1),(a,b-1),(a+1,b-1)\}$, and $\{(a-1,b+1),(a,b+1),(a+1,b+1)\}$. Henceforth, if v has six neighbours in U_2 and thus only two in $U_{\geq 3}$, then those two neighbours in $U_{\geq 3}$ are either $(a-1,b-1)$ and $(a+1,b+1)$ or $(a-1,b+1)$ and $(a+1,b-1)$.

Assume $(a-1,b-1)$ and $(a+1,b+1)$ are the sole neighbours of v in $U_{\geq 3}$. Then $(a-1,b)$ and $(a-1,b+1)$ are in U_2 and have distinct identifiers, so the identifier of $(a-1,b)$ is $\{v,(a-2,b-1)\}$. In particular, $(a-2,b-1)\in C$. In the same way, we have that $(a-1,b-2)$, $(a+2,b+1)$, and $(a+1,b+2)$ are in C.

Similarly, if $(a-1,b+1)$ and $(a+1,b-1)$ are the sole neighbours of v in $U_{\geq 3}$, we have that $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\}\subseteq C$. $\Diamond$

A *defective vertex* is a vertex in C_1 with six neighbours in U_2. Let $v=(a,b)$ be a defective vertex. The *team* of v is one of the two sets $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\}$ and $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\}$ which is included in C. By Claim 1.4, the team exists. Moreover, by Claim 1.1, at least two vertices of the team are in $C_{\geq 3}$. Those vertices are the *partners* of v (that is, the vertices of the team that are in $C_{\geq 3}$).

Observe that a full C-vertex is partner of at most two defective vertices and a side C-vertex is partner of at most one defective vertex (see Figure 2 and recall that a partner is in $C_{\geq 3}$).

We should apply the following discharging rules.

(R1) Every C-vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i.

(R2) Every defective vertex receives $\frac{1}{54}$ from each of its partners.

We shall now prove that the final charge $\mathrm{chrg}(v)$ of every vertex v is at least $2/9$. We distinguish several cases depending on the set to which v belongs.

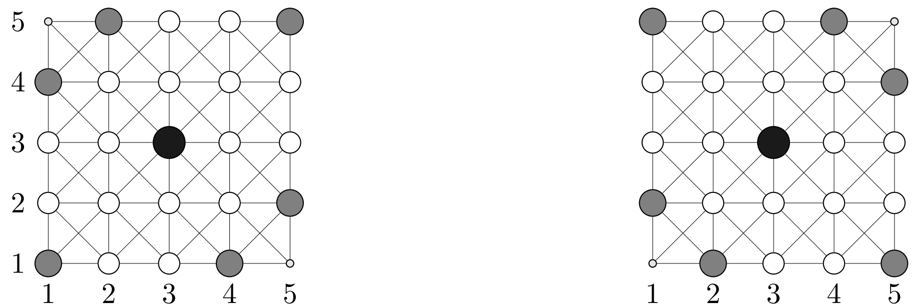

Figure 2: The configuration of a defective vertex: $v = (a, b) = (3, 3)$.

- Assume first that $v \in U$. There is some i such that $v \in U_i$. Then v receives $\frac{2}{9i}$ from each of its i neighbours in C by (R1). Thus $\text{chrg}(v) = 2/9$.

- Assume that v is a side C-vertex. By Claims 1.4 and 1.3(i), either it is in C_1 and it has three or five neighbours in $U_{\geq 2}$, or it is in $C_{\geq 2}$ and it has two or four neighbours, with at most one in U_1. In both cases, v sends at most $\frac{5}{9}$ by (R1). Moreover, it is partner of at most one vertex, so it sends at most $\frac{1}{54}$ by (R2). Hence $\text{chrg}(v) \geq 1 - \frac{5}{9} - \frac{1}{54} = \frac{23}{54} > \frac{2}{9}$.

- Assume that v is a full $C_{\geq 4}$-vertex. It has at most five U-neighbours with at most one in U_1 by Claim 1.3(i). Henceforth it gives at most $\frac{2}{9}(1 + 4 \times \frac{1}{2}) = 2/3$ by (R1). Moreover, v is partner of at most two defective vertices, hence it sends at most $\frac{1}{27}$ by (R2). Hence $\text{chrg}(v) \geq 1 - \frac{2}{3} - \frac{1}{27} = \frac{8}{27} > \frac{2}{9}$.

- Assume that v is a full C_3-vertex. It has six U-neighbours with at most one in U_1 and at least one in $U_{\geq 3}$ by Claims 1.3(i) and 1.3(iii). Therefore it gives at most $\frac{2}{9}(1+4\times\frac{1}{2}+\frac{1}{3}) = \frac{20}{27}$. Moreover, v is the partner of at most two defective vertices, hence it sends at most $\frac{1}{27}$ by (R2). Hence $\text{chrg}(v) \geq 1 - \frac{20}{27} - \frac{1}{27} = \frac{7}{27} > \frac{2}{9}$.

- Assume that v is a full C_2-vertex. It has seven U-neighbours with at most one in U_1 and at least three in $U_{\geq 3}$ by Claim 1.3. Therefore it sends at most $\frac{2}{9}(1 + 3 \times \frac{1}{2} + 3 \times \frac{1}{3}) = \frac{7}{9}$. Moreover, it sends nothing by (R2), because partners are in $C_{\geq 3}$ by definition. Hence $\text{chrg}(v) \geq 1 - \frac{7}{9} = \frac{2}{9}$.

- Assume that v is a full C_1-vertex. If v is not defective, then by Claim 1.4, it has no U_1-neighbour and at most five U_2-neighbours. Thus it sends at most $\frac{2}{9}(5 \times \frac{1}{2} + 3 \times \frac{1}{3}) = \frac{7}{9}$ by (R1). Moreover its sends nothing by (R2) because partners are in $C_{\geq 3}$. Hence $\text{chrg}(v) \geq 1 - \frac{7}{9} = \frac{2}{9}$.

  If v is defective, then it has no U_1-neighbour and at most six U_2-neighbours. Thus it sends at most $\frac{2}{9}(6 \times \frac{1}{2} + 2 \times \frac{1}{3}) = \frac{22}{27}$ by (R1). Moreover, by (R2) it receives $\frac{1}{54}$ from each of its partners, so at least $\frac{1}{27}$ in total because it has at least two partners. Hence $\text{chrg}(v) \leq 1 - \frac{22}{27} + \frac{1}{27} = \frac{2}{9}$.

□

The bound 2/9 of Theorem 1 is best possible because of the infinite king grid $\mathcal{G}_K$. From C_∞, one can also easily derive identifying codes of $\mathcal{H}_K$ and $\mathcal{Q}_K$ with density 2/9. However, for the other king grids, one can prove better that the density is larger than $\frac{2}{9}$.

The idea is to prove that there are vertices whose final charge is greater than 2/9. For every vertex v, its *excess* is $\mathrm{exc}(v) = \mathrm{chrg}(v) - 2/9$. For a set X of vertices, its *excess* is $\mathrm{exc}(X) = \sum_{x \in X} \mathrm{exc}(x)$. We shall prove that some vertices have positive excess.

Theorem 2. *If G is a finite king grid, then $d^*(G) > \frac{2}{9}$.*

Proof. Let $G = P_\ell \boxtimes P_k$ be a finite grid. If $k = 1$ or $\ell = 1$, then one easily sees that $d^*(G) \geq \frac{1}{2}$ and if $k = 2$ or $\ell = 2$, then G has no identifying code, so $d^*(G) = +\infty$.

Suppose now that $k, \ell \geq 3$. We proceed as in the proof of Theorem 1. Let C be an identifying code of G. We start with the same initial charge and apply the same discharging rules. After applying them, every vertex v has a charge $\mathrm{chrg}(v)$ which is at least 2/9. It suffices to prove that a vertex has positive excess at least ϵ for some fixed ϵ. This would imply $d^*(G) \geq \frac{2}{9} + \frac{\epsilon}{k \cdot \ell}$.

To do so we shall prove that there is a side C-vertex or a $C_{\geq 3}$-vertex. Such a vertex has excess at least $\frac{1}{27}$ as shown in the proof of Theorem 1.

Suppose for a contradiction that there is no such vertex. Then $(1,1)$, $(1,2)$, $(1,3)$ and $(2,1)$ are not in C. Therefore $(2,2)$ is in C because $C[(1,1)] \neq \emptyset$. Moreover, $(2,2)$ has a neighbour w in C, because $C[(1,1)] \neq C[(2,2)]$. Now by Claim 1.1, one vertex among $(2,2)$ and w is in $C_{\geq 3}$, a contradiction. $\square$

3 King strips with at least seven rows

The aim of this section is to give better lower bound than 2/9 on the density of identifying codes in king strips of large height. We need some definitions. The b-th row of $\mathcal{K}_k$ is $R_b = \{(a,b) \ : \ a \in \mathbb{Z}\}$. The a-th column of $\mathcal{K}_k$ is $Q_a^k = \{(a,b) \ : \ 1 \leq b \leq k\}$. In the following, we always omit k and just right Q_a for the a-th column since k is always fixed and there is no risk of confusion. The *bottom* of $\mathcal{K}_k$ is $B = R_1 \cup R_2 \cup R_3$, and its *top* is $T = R_k \cup R_{k-1} \cup R_{k-2}$. For every integer a, we set $B[a] = B \cap (Q_{a-1} \cup Q_a \cup Q_{a+1})$ and $T[a] = T \cap (Q_{a-1} \cup Q_a \cup Q_{a+1})$.

Theorem 3. *For every $k \geq 7$, $d^*(\mathcal{K}_k) \ \geq \ \frac{2}{9} + \frac{8}{81k}$.*

Proof. We proceed as in the proof of Theorem 1. Fix $k \geq 7$. Let C be an identifying code of $\mathcal{K}_k$. We start with the same initial charge and apply the same discharging rules. After applying them, every vertex v has a charge $\mathrm{chrg}(v)$ which is at least 2/9 and so it has non-negative excess.

We shall prove that many vertices in the bottom (and top) of $\mathcal{K}_k$ have positive excess. The following claim is easy.

Claim 3.1. *A vertex in B is the partner of at most one defective vertex.*

This claim implies that the lower bounds on the excess of $C_{\geq 3}$-vertices given in Theorem 1 can be increased by $\frac{1}{54}$ for such vertices in B because those bounds were considering that a vertex could send $\frac{1}{27}$ by (R2), while it sends at most $\frac{1}{54}$. Consequently, we obtain the following.

Claim 3.2. *.*

(i) Every C_3-vertex in B has excess at least $\frac{1}{18}$.

(ii) Every C_4-vertex in B has excess at least $\frac{5}{54}$.

Claim 3.3. *Every vertex in R_2 has a C-neighbour in R_3.*

Proof. The closed neighbourhood of $(a,1)$ is included in the closed neighbourhood of $(a,2)$. The identifiers of those two vertices are distinct, thus there is a C-vertex in $C[(a,2)] \setminus C[(a,1)]$ which necessarily is in R_3. $\Diamond$

Claim 3.4. *If v is a side C-vertex, then* $\mathrm{exc}(v) \geq \frac{2}{9}$.

Proof. Let $v = (a,1)$ be a side vertex.

If v is in C_1, then it has five U-neighbours with none in U_1 by Claim 1.4. Hence, it gives at most $\frac{5}{9}$ by (R1). So $\mathrm{exc}(v) \geq \frac{7}{9} - \frac{5}{9} = \frac{2}{9}$.

If v is in $C_{\geq 3}$, then it has at most three U-neighbours with at most one in U_1 by Claim 1.3(i). Hence it sends at most $\frac{4}{9}$ by Rule 1. Moreover, it is partner of at most one vertex, so it sends at most $\frac{1}{54}$ by (R2). Thus $\mathrm{exc}(v) \geq \frac{7}{9} - \frac{4}{9} - \frac{1}{54} > \frac{2}{9}$.

If v is in C_2, then it has at least two U-neighbours adjacent to its C-neighbour, and which consequently are in $U_{\geq 3}$. Therefore v sends at most $\frac{2}{9}(1 + \frac{1}{2} + 2 \times \frac{1}{3}) = \frac{13}{27}$. Thus $\mathrm{exc}(v) \geq \frac{7}{9} - \frac{13}{27} = \frac{8}{27} > \frac{2}{9}$. $\Diamond$

Claim 3.5. *If $(a+j,1) \in U$ for all $j \in \{-2,-1,0,1,2\}$, then* $\mathrm{exc}(B[a]) \geq \frac{4}{27}$.

Proof. Let a such that $(a+j,1) \in U$ for all $j \in \{-2,-1,0,1,2\}$.

Assume first that $(a,2) \in C$.

If $(a-1,2) \in C$, then By Claim 3.3, both $(a-1,2)$ and $(a,2)$ have a C-neighbour in R_3. Moreover, since $C[(a,1)] \neq C[(a-1,1)]$, $(a-2,2)$ or $(a+1,2)$ is in C. If $(a-1,2) \in C$, then a vertex in $\{(a-1,2),(a,2)\}$ is in $C_{\geq 4}$ and the other in $C_{\geq 3}$. Thus, by Claim 3.2, $\mathrm{exc}(B[a]) \geq \frac{5}{54} + \frac{1}{18} = \frac{4}{27}$. Henceforth we may suppose that $(a-1,2) \notin C$. By symmetry, we may also suppose that $(a+1,2) \notin C$. Hence, since the identifiers of $(a-1,1)$ and $(a+1,1)$ are distinct from $C[(a,1)] = \{(a,2)\}$, necessarily $(a-2,2)$ and $(a+2,2)$ are in C.

Suppose that $(a-1,3) \in C$. This vertex has a C-neighbour in R_4, because its identifier is distinct from the one of $(a-1,2)$. Moreover $(a-1,2) \in U_{\geq 3}$ and $(a,3)$ is either in C or in $U_{\geq 3}$ because it has a C-neighbour in R_4, because its identifier is distinct from the one of $(a,2)$. In addition, $(a-1,3)$ is partner of at most one vertex, so it sends at most $\frac{1}{54}$ by (R2). Thus $\mathrm{exc}(B[a]) \geq \mathrm{exc}((a-1,3)) \geq \frac{7}{9} - \frac{2}{9}(1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{3}) - \frac{1}{54} = \frac{1}{6}$. Henceforth, we may suppose that $(a-1,3) \notin C$, and by symmetry $(a+1,3) \notin C$. Thus, by Claim 3.3, $(a,3) \in C$.

Observe that $(a-1,3)$, $(a,3)$ and $(a+1,3)$ have a neighbour in R_4 because their identifiers are distinct from those of $(a-1,2)$, $(a,2)$ and $(a+1,2)$ respectively. In particular, $(a-1,3)$ and $(a+1,3)$ are in $U_{\geq 4}$. Thus $(a,3)$ is in $C_{\geq 3}$, has two neighbours in $U_{\geq 3}$ ($(a-1,2)$ and $(a+1,2)$) and two neighbours in $U_{\geq 4}$ ($(a-1,3)$ and $(a+1,3)$), is partner of at most one vertex. Hence $\mathrm{exc}(B[a]) \geq \mathrm{exc}((a,3)) \geq \frac{7}{9} - \frac{2}{9}(1 + \frac{1}{2} + 2 \times \frac{1}{3} + 2 \times \frac{1}{4}) - \frac{1}{54} = \frac{1}{6}$.

Assume now that $(a,2) \notin C$. Since $(a,1)$ is adjacent to a neighbour in C, we may assume by symmetry that $(a-1,2) \in C$. We distinguish two subcases depending on whether $(a+1,2)$ is in C or not.

- Suppose that $(a+1,2) \in C$.

  Notice that $(a,3)$ must have a C-neighbour in R_4, because its identifier is distinct from the one of $(a,2)$. Thus if $(a,3) \in C$, then $\mathrm{exc}((a,3)) \geq \frac{5}{54}$ by Claim 3.2(ii). If $(a-1,2)$ or $(a+1,2)$ is in $C_{\geq 3}$, then it has excess at least $\frac{1}{18}$ by Claim 3.2(i), and $\mathrm{exc}(B[a]) \geq \frac{5}{54} + \frac{1}{18} = \frac{4}{27}$. Henceforth we may suppose that both $(a-1,2)$ and $(a+1,2)$ are in C_2. This implies that $(a-1,3)$ and $(a+1,3)$ are in $U_{\geq 3}$ because their identifiers are distinct from those of $(a-1,2)$ and $(a+1,2)$. Therefore, $\mathrm{exc}((a,3)) \geq \frac{7}{9} - \frac{2}{9}(1 + \frac{1}{2} + 3 \times \frac{1}{3}) - \frac{1}{54} = \frac{11}{54} \geq \frac{4}{27}$. If $(a,3) \notin C$, then w.l.g. $(a-1,3) \in C$ and $\mathrm{exc}((a-1,3)) \geq \frac{7}{9} - \frac{2}{9}(1 + \frac{1}{2} + 2 \times \frac{1}{3} + +2 \times \frac{1}{4}) - \frac{1}{54} = \frac{1}{6} \geq \frac{4}{27}$.

- Suppose that $(a+1,2) \notin C$. Then $(a+2,2) \in C$ because $(a+1,1)$ has a C-neighbour and $(a-2,2) \in C$ because $C[(a-1,1)] \neq C[(a,1)]$. In particular $(a-1,2) \in C_{\geq 3}$ because it has a neighbour in R_3 by Claim 3.3.

  If $(a-1,3) \in C$, then it is in $C_{\geq 4}$ because it has a neighbour in R_4 to have its identifier distinct from the one of $(a-1,2)$. Hence, by Claim 3.2, $\mathrm{exc}(B[a]) \geq \frac{5}{54} + \frac{1}{18} = \frac{4}{27}$. Henceforth, we may suppose that $(a-1,3) \in U$.

  Observe that $(a,1) \in U_1$, $(a-1,1) \in U_2$ and $(a-2,1) \in U_{\geq 3}$ because its identifier is distinct from $\{(a-2,2,a-1,2)\} = C[(a-1,1)]$.

  If $(a,3) \in C$, then $(a,2) \in U_{\geq 2}$, $(a-1,3) \in U_{\geq 4}$ because it has a neighbour in R_4 to have its identifier distinct from the one of $(a-1,2)$ and $(a-2,3) \in U_{\geq 3} \cup C$ because its identifier is distinct from $\{(a-2,2,a-1,2)\} = C[(a-1,1)]$. Henceforth $\mathrm{exc}((a-1,2)) \geq \frac{7}{9} - \frac{2}{9}(1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{3} + \frac{1}{4}) - \frac{1}{54} = \frac{1}{9}$. Moreover $(a,3)$ has neighbour in R_4 to have its identifier distinct from the one of $(a,2)$, therefore $(a,3) \in C_{\geq 3}$ and so its excess is at least $\frac{1}{18}$ by Claim 3.2(i). Hence $\mathrm{exc}(B[a]) \geq \frac{1}{9} + \frac{1}{18} = \frac{1}{6}$. Henceforth, we may suppose that $(a,3) \in U$, and so $(a-2,3) \in C$.

  Necessarily, $(a+1,3) \in C$ because $C[(a,2)] \neq C[(a,1)]$. Furthermore, $(a+1,3)$ has a neighbour in R_4 to have its identifier distinct from the one of $(a+1,2)$. Thus $(a+1,3) \in C_{\geq 3}$ and its excess is at least $\frac{1}{18}$ by Claim 3.2. Then $(a-1,3) \in U_{\geq 4}$ because it has a neighbour in R_4 to have its identifier distinct from the one of $(a-1,2)$ and $(a,3) \in U_{\geq 3}$ because it has a neighbour in R_4 to have its identifier distinct from the one of $(a,2)$. Moreover, $(a-2,1) \in U_{\geq 3}$ since it has an identifier distinct from the one of $(a-1,1)$. Therefore $\mathrm{exc}((a-1,2)) \geq \frac{7}{9} - \frac{2}{9}(1 + 2 \times \frac{1}{2} + 2 \times \frac{1}{3} + \frac{1}{4}) - \frac{1}{54} = \frac{1}{9}$. Hence $\mathrm{exc}(B[a]) \geq \frac{1}{9} + \frac{1}{18} = \frac{1}{6}$.

$\Diamond$

Claim 3.6. *If $(a-1,1)$, $(a,1)$ and $(a+1,1)$ are in U, then $\mathrm{exc}(B[a]) \geq \frac{2}{27}$.*

Proof. Assume $(a-1,1)$, $(a,1)$ and $(a+1,1)$ are in U.

If $B[a]$ contains a vertex of $C_{\geq 4}$, then by Claim 3.2, $\mathrm{exc}(B[a]) \geq \frac{5}{54}$, therefore we may assume that all C-vertices of $B[a]$ are in $C_{\leq 3}$. If $B[a]$ contains two vertices of C_3, then by Claim 3.2, $\mathrm{exc}(B[a]) \geq 2 \times \frac{1}{18} = \frac{1}{9}$. So we may assume that $B[a]$ has at most one vertex in C_3.

Assume first that $(a,2) \in C$. By Claim 3.3 it has a C-neighbour w in R_3. By symmetry, we may assume that $w \in \{(a-1,3),(a,3)\}$. By Claim 3.3, there is a C-vertex in $\{(a,3),(a+1,3),)(a+2,3)\}$, and there must be a C-vertex in $\{(a-1,4),(a,4),)(a+1,4)\}$ because

$C[(a,3)] \neq C[(a,2)]$. Consequently, w and $(a,2)$ cannot be partner of any vertex. Thus the vertex in C_3 among those two vertices has excess $\frac{1}{18} + \frac{1}{54} = \frac{2}{27}$. Henceforth, we may assume $(a,2) \notin C$.

Since $C[(a,1)] \neq \emptyset$, one of $(a+1,2)$ and $(a-1,2)$ is in C. By symmetry, we may assume $(a+1,2) \in C$. Then $(a,3)$ has a C-neighbour in R_4, because $C[(a,3)] \neq C[(a,2)]$. If $(a,3)$ is in C, then it is in C_3, so $(a+1,2) \in C_2$ because $B[a]$ has at most one vertex in C_3. Thus $(a-1,2)$, $(a+2,1)$ and $(a+2,2)$ are in U, so $C[(a,1)] = C[(a+1,1)] = \{(a+1,2)\}$, a contradiction. Hence $(a,3) \in U$.

Assume that $(a+1,3)$ is in C. Since $B[a]$ contains at most one vertex in C_3, then $(a+2,2)$ and $(a+2,3)$ are in U. Furthermore, one of $\{(a+1,2),(a+1,3)\}$, say t, is in in C_3, and the other is in C_2 and has identifier $\{(a+1,2),(a+1,3)\}$. Hence the vertices $(a,2)$, $(a,3)$, $(a+2,2)$ and $(a+2,3)$ are in $U_{\geq 3}$ because their identifiers contain and are different from $\{(a+1,2),(a+1,3)\}$. Hence $\mathrm{exc}(B[a]) \geq \mathrm{exc}(t) \geq \frac{7}{9} - \frac{2}{9}(1 + \frac{1}{2} + 4 \times \frac{1}{3}) - \frac{1}{54} = \frac{7}{54}$.

Henceforth, we may assume $(a+1,3) \in U$. If $(a-1,2) \in C$, then symmetrically to the preceding argument, a vertex t in $\{(a-1,2),(a-1,3)\}$ is in C_3 and $\mathrm{exc}(B[a]) \geq \mathrm{exc}(t) = \frac{7}{54}$. Henceforth, we may assume $(a-1,2) \in U$. Now, $(a+2,1)$ or $(a+2,2)$ is in C because $C[(a+1,1)] \neq C[(a,1)]$, and $(a+2,3)$ is in C because $C[(a+1,2)] \neq C[(a+1,1)]$. Hence $(a+1,2)$ is in C_3. In addition, it is the partner of no vertex because $(a-1,3)$ and $(a+2,3)$ are in C. Thus $\mathrm{exc}(B[a]) \geq \mathrm{exc}((a+1,2)) \geq \frac{1}{18} + \frac{1}{54} = \frac{2}{27}$. $\diamondsuit$

We first apply the following discharging rule.

(R3) Every side C-vertex sends $\frac{2}{27}$ to each of its two side neighbours.

Let us denote by exc_3 the excess after applying (R3).

Claim 3.7. $\mathrm{exc}_3(B[a]) \geq \frac{4}{27}$ *for every integer a.*

Proof. If $(a,1) \in C$, then $\mathrm{exc}_3(B[a]) \geq \mathrm{exc}((a,1)) \geq \frac{2}{9}$ by Claim 3.4. If $(a-1,1) \in C$, then $\mathrm{exc}_3(B[a]) \geq \mathrm{exc}_3((a-1,1)) + \mathrm{exc}_3((a,1)) = \mathrm{exc}((a-1,1)) - 2 \times \frac{2}{27} + \frac{2}{27} \geq \frac{4}{27}$ by Claim 3.4. Similarly, if $(a+1,1) \in C$, then $\mathrm{exc}_3(B[a]) \geq \frac{4}{27}$.

Henceforth, we assume that $(a-1,1),(a,1),(a+1,1)$ are in U. If $(a-2,1)$ and $(a+2,1)$ are also in U, then $\mathrm{exc}_3(B[a]) = \mathrm{exc}(B[a]) \geq \frac{4}{27}$ by Claim 3.5. If not, say $(a-2,1) \in C$, then $\mathrm{exc}_3(B[a]) \geq \mathrm{exc}(B[a]) + \frac{2}{27}$ because $(a-1,1)$ receives $\frac{2}{27}$ from $(a-2,1)$ by (R3). Moreover, $\mathrm{exc}(B[a]) \geq \frac{2}{27}$ by Claim 3.6, so $\mathrm{exc}_3(B[a]) \geq \frac{4}{27}$. $\diamondsuit$

We then apply the following rule

(R4) For every integer a, every vertex v in $B[a]$ sends $\frac{1}{3k}\mathrm{exc}_3(v)$ to each vertex in Q_a, and every vertex in $T[a]$ sends $\frac{1}{3k}\mathrm{exc}_3(v)$ to each vertex in Q_a.

Let $\mathrm{exc}_4(v)$ be the excess of every vertex after (R4). Since $k \geq 7$, a vertex $v = (a,b)$ cannot be both in the bottom and the top. Moreover it belongs only to $B[a']$ for $a' \in \{a-1,a,a+1\}$. Hence v sends to at most three columns by (R4). Therefore it sends at most $\mathrm{exc}_3(v)$. Moreover every vertex v receives $\frac{1}{3k}(\mathrm{exc}_3(B[a]) + \mathrm{exc}_3(T[a]))$. By Claim 3.7 and symmetry, $\mathrm{exc}_3(B[a]) \geq \frac{4}{27}$ and $\mathrm{exc}_3(T[a]) \geq \frac{4}{27}$. Hence $\mathrm{exc}_4(v) \geq \mathrm{exc}_3(v) - \mathrm{exc}_3(v) + \frac{1}{3k} \cdot \frac{8}{27} = \frac{8}{81k}$.

Thus $d(C,\mathcal{K}_k) \geq \frac{2}{9} + \frac{8}{81k}$. $\square$

Theorem 4. *For every $k \geq 7$,*

$$d^*(\mathcal{K}_k) \;\leq\; \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \textit{if } k \equiv 0 \mod 3, \\ \frac{2}{9} + \frac{8}{18k}, & \textit{if } k \equiv 1 \mod 3, \\ \frac{2}{9} + \frac{7}{18k}, & \textit{if } k \equiv 2 \mod 3. \end{cases}$$

Proof. Recall the identifying code C_∞ of the infinite king grid in Figure 1:

$$\begin{aligned} C_\infty \;=\; \Big\{&(6a+2,6b+1),(6a+4,6b+2),(6a+6,6b+2),(6a+2,6b+3),\\ &(6a+5,6b+4),(6a+1,6b+5),(6a+3,6b+5),(6a+5,6b+6) \;\Big|\; a,b \in \mathbb{Z}\Big\}. \end{aligned}$$

Let $k \geq 7$ be an integer. If $k \equiv 0 \mod 3$ or $k \equiv 2 \mod 3$, let

$$C'_k \;=\; \Big(C_\infty \;\cap\; \mathbb{Z}\times[k]\Big) \;\cup\; \Big\{(6a+2,3),(6a+5,3),(6a+2,k-2),(6a+5,k-2) \;\Big|\; a \in \mathbb{Z}\Big\};$$

if $k \equiv 1 \mod 3$, let

$$C'_k \;=\; \Big(C_\infty \;\cap\; \mathbb{Z}\times\{2,\ldots,k+1\}\Big) \;\cup\; \Big\{(6a+2,4),(6a+5,4),(6a+2,k-1),(6a+5,k-1) \;\Big|\; a \in \mathbb{Z}\Big\}.$$

One can easily check that C'_k is an identifying code of $\mathcal{K}_k$ when $k \equiv 0 \mod 3$ or $k \equiv 2 \mod 3$, and that C'_k is an identifying code of the strip induced by the rows 2 to $k+1$ (which is isomorphic to $\mathcal{K}_k$). For an example, C'_5 and C'_6 are the second identifying codes of Figures 3 and 4, respectively (new vertices in black). It is not hard to verify that

$$d(C'_k, \mathcal{K}_k) \;=\; \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \mod 3, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \mod 3, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

□

4 King strips with three, four, five or six rows

Given an integer $k \geq 3$ and an identifying code C of $\mathcal{K}_k$, let $d(C, R_i)$ denote the density of C in the row i:

$$d(C, R_i) \;=\; \limsup_{n\to\infty} \left(\frac{1}{2n+1}\right) \cdot \Big| C \cap \{(a,i) \;:\; a \in \mathbb{Z}, |a| \leq n\} \Big|.$$

Notice that $d(C, \mathcal{K}_k) = \frac{1}{k}\sum_{i=1}^{k} d(C, R_i)$. In order to prove the lower bounds, we first prove the following auxiliary lemma.

Lemma 5. *Let $k \geq 3$ be an integer and let C be an identifying code of $\mathcal{K}_k$. Then $d(C,R_1) + d(C,R_2) \geq 1/2$, $d(C,R_k) + d(C,R_{k-1}) \geq 1/2$, $d(C,R_3) \geq 1/3$ and $d(C,R_{k-2}) \geq 1/3$.*

Proof. For every $a \in \mathbb{Z}$, $C \cap \{(a-1,3),(a,3),(a+1,3)\} \neq \emptyset$, since, otherwise, $(a,1)$ and $(a,2)$ have the same identifier, a contradiction. Then, $d(C,R_3) \geq 1/3$. Symmetrically, we have $d(C,R_{k-2}) \geq 1/3$.

Consider now $R_1 \cup R_2$. Set $A = \{a \in \mathbb{Z} \ : \ C \cap \{(a,1),(a,2)\} = \emptyset\}$, and $B = \mathbb{Z} \setminus A$. For each $a \in A$ let $s(a)$ be defined as follows. If $a-1 \in B$, then $s(a) = a-1$, otherwise $s(a) = a-3$. We now prove that s is an injective mapping from A into B. If $s(a) = a-1$, then $s(a) \in B$ by definition. If not, then $C \cap \{(a-1,1),(a-1,2)\} = \emptyset$. Hence either $(a-2,1) \in C$ or $(a-2,2) \in C$ because $C[(a-1,1)] \neq \emptyset$, which implies that $a-2 \in B$. Moreover, since $C[(a-2,1)] \neq C[(a-1,1)]$, then either $(a-3,1) \in C$ or $(a-3,2) \in C$, so $s(a) = a-3$ is in B. Then s is injective. Therefore $d(C,R_1) + d(C,R_2) \geq 1/2$. Symmetrically, we have $d(C,R_k) + d(C,R_{k-1}) \geq 1/2$. □

Theorem 6. $d^*(\mathcal{K}_5) = 4/15 = 0.2666\ldots$.

Proof. Let C be an identifying code of $\mathcal{K}_5$. From Lemma 5, we have $d(C,R_1)+d(C,R_2) \geq 1/2$, $d(C,R_4) + d(C,R_5) \geq 1/2$ and $d(C,R_3) \geq 1/3$. Thus $d(C,\mathcal{K}_5) = \frac{1}{5}\sum_{i=1}^{5} d(C,R_i) \geq \frac{1}{5}(\frac{1}{2} + \frac{1}{3} + \frac{1}{2}) = 4/15$. The two periodic identifying codes of $\mathcal{K}_5$ generated by the tiles depicted in Figure 3 have density 4/15. Thus $d^*(\mathcal{K}_5) = 4/15$. □

Figure 3: Two tiles generating optimal identifying codes of $\mathcal{K}_5$ (density 4/15)

Theorem 7. $d^*(\mathcal{K}_6) = 5/18 = 0.2777\ldots$.

Proof. Let C be an identifying code of $\mathcal{K}_6$. From Lemma 5, we have $d(C,R_1) + d(C,R_2) \geq 1/2$, $d(C,R_5) + d(C,R_6) \geq 1/2$, $d(C,R_3) \geq 1/3$ and $d(C,R_4) \geq 1/3$. Thus $d(C,\mathcal{K}_6) = \frac{1}{6}\sum_{i=1}^{6} d(C,R_i) \geq \frac{1}{6}(\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2}) = 5/18$. The two periodic identifying codes of $\mathcal{K}_6$ generated by the tiles depicted in Figure 4 have density 5/18. Thus $d^*(\mathcal{K}_6) = 5/18$. □

The proofs of the following lemmas of this section use the Discharging Method on the columns. Recall that, with k fixed, Q_a is the a-th column of $\mathcal{K}_k$ for every $a \in \mathbb{Z}$. The general idea is to consider any identifying code C of $\mathcal{K}_k$ and associate to every $a \in \mathbb{Z}$ the initial charge $\mathrm{chrg}_0(a) = |Q_a \cap C|$. We will say that $a \in \mathbb{Z}$ is *unsatisfied* if $\mathrm{chrg}(a)$ is less than a given value q, and *satisfied* otherwise. Then we apply some local discharging rules (from satisfied a's to unsatisfied a's). Here local means that there is no charge transfer from a satisfied $a' \in \mathbb{Z}$ to an unsatisfied $a \in \mathbb{Z}$ with $|a' - a|$ greater than a given constant. Finally, we prove that, after the discharging, every integer $a \in \mathbb{Z}$ is satisfied. With this, we obtain that the density of C is at least q/k.

Theorem 8. $d^*(\mathcal{K}_3) = 1/3$.

Figure 4: Two tiles generating optimal identifying codes of $\mathcal{K}_6$ (density 5/18)

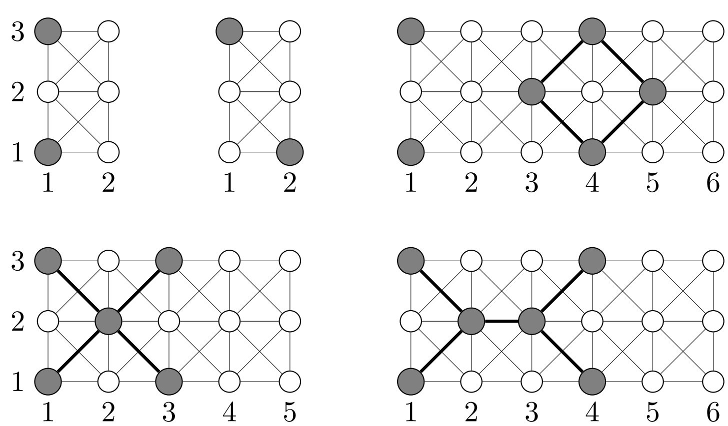

Figure 5: Five tiles generating optimal identifying codes of $\mathcal{K}_3$ (density 1/3)

Proof. The five periodic identifying codes of $\mathcal{K}_3$ generated by the tiles depicted in Figure 5 have density 1/3. Thus $d^*(\mathcal{K}_3) \leq 1/3$.

Let us now prove $d^*(\mathcal{K}_3) \geq 1/3$. Let C be an identifying code of $\mathcal{K}_3$. For every $a \in \mathbb{Z}$, let the initial charge $\mathrm{chrg}_0(a)$ be $|Q_a \cap C|$. We say that $a \in \mathbb{Z}$ is *satisfied* if its current charge is at least 1, and *unsatisfied* otherwise. We apply the following five discharging rules, Rule i for $i = 1$ to 5, one after another. We denote by $\mathrm{chrg}_i(a)$ the charge of a after applying Rule i.

> **[Rule** $1 \leq i \leq 5$**]**: every unsatisfied $a \in \mathbb{Z}$ receives charge 1 from $a-i$, if $\mathrm{chrg}_{i-1}(a-i) \geq 2$.

We shall prove that after applying the discharging rules, every $a \in \mathbb{Z}$ is satisfied. This yields $d(C, \mathcal{K}_3) \geq 1/3$. Observe that once an integer becomes satisfied, then its remains satisfied.

Consider an initially unsatisfied integer a. Then, $Q_a \cap C = \emptyset$. Assume for a contradiction that a is unsatisfied after applying the five discharging rules.

Then $|Q_{a-1} \cap C| < 2$, for otherwise it is satisfied after Rule 1. Therefore, by symmetry, we may assume that we are in one of the following cases: $Q_{a-1} \cap C = \{(a-1,2)\}$ (Case 1), $Q_{a-1} \cap C = \{(a-1,1)\}$ (Case 2) and $Q_{a-1} \cap C = \emptyset$ (Case 3).

Case 1: $Q_{a-1} \cap C = \{(a-1,2)\}$. Since $C[(a-1,1)] \neq C[(a-1,2)]$ and $C[(a-1,3)] \neq C[(a-1,2)]$, then $(a-2,1),(a-2,3) \in C$, hence a receives 1 from $a-2$ by Rule 2 and is satisfied, a contradiction.

Case 2: $Q_{a-1} \cap C = \{(a-1,1)\}$. Since $C[(a-1,1)] \neq C[(a-1,2)]$, then $(a-2,3) \in C$. $|Q_{a-2} \cap C| \leq 1$, for otherwise a would receive 1 from $a-2$ by Rule 2 and be satisfied. So $(a-2,1),(a-2,2) \notin C$. Since $C[(a-2,1)] \neq C[(a-1,1)] = \{(a-1,1)\}$, then $C \cap \{(a-3,1),(a-3,2)\} \neq \emptyset$. Moreover, since $C[(a-2,3)] \neq C[(a-1,3)] = \{(a-2,3)\}$, then $C \cap \{(a-3,3),(a-3,2)\} \neq \emptyset$. In addition, $|Q_{a-3} \cap C| \leq 1$, for otherwise a would receive 1 from $a-3$ by Rule 3 and be satisfied. Thus $Q_{a-3} \cap C = \{(a-3,2)\}$. Therefore $(a-4,1) \in C$, since $C[(a-3,2)] \neq C[(a-3,3)]$. Moreover, $\{(a-4,2),(a-4,3)\} \cap C \neq \emptyset$, since $C[(a-3,3)] \neq C[(a-2,3)] = \{(a-2,3),(a-3,2)\}$. Thus $|Q(a-4) \cap C| \geq 2$. Hence a receives 1 from $a-4$ by Rule 4, and is satisfied, a contradiction.

Case 3: $Q_{a-1} \cap C = \emptyset$. Since $C[(a-1,1)] \neq C[(a-1,2)]$ and $C[(a-1,3)] \neq C[(a-1,2)]$, then $(a-2,1),(a-2,3) \in C$. Thus $a-1$ receives 1 from $a-2$ by Rule 1 and is satisfied. Now $(a-2,2) \notin C$, for otherwise a would receive 1 from $a-2$ by Rule 2 and be satisfied. Since $C[(a-2,1)] \neq C[(a-1,1)] = \{(a-2,1)\}$, then $C \cap \{(a-3,1),(a-3,2)\} \neq \emptyset$. Moreover, since $C[(a-2,3)] \neq C[(a-1,3)] = \{(a-2,3)\}$, then $C \cap \{(a-3,3),(a-3,2)\} \neq \emptyset$. In addition, $|Q_{a-3} \cap C| \leq 1$, for otherwise a would receive 1 from $a-3$ by Rule 3 and be satisfied. Thus $Q_{a-3} \cap C = \{(a-3,2)\}$. Since $C[(a-3,1)] \neq C[(a-2,1)]$, then $C \cap \{(a-4,1),(a-4,2)\} \neq \emptyset$. Moreover, since $C[(a-3,3)] \neq C[(a-2,3)]$, then $C \cap \{(a-4,3),(a-4,2)\} \neq \emptyset$. In addition, $|Q_{a-4} \cap C| \leq 1$, for otherwise a would receive 1 from $a-4$ by Rule 4 and be satisfied. Thus $Q_{a-4} \cap C = \{(a-4,2)\}$. Since $C[(a-4,1)] \neq C[(a-4,2)]$ and $C[(a-4,3)] \neq C[(a-4,2)]$, then $(a-5,1),(a-5,3) \in C$. Hence a receives 1 from $a-5$ by Rule 5 and is satisfied, a contradiction. □

Theorem 9. $d^*(\mathcal{K}_4) = 5/16$.

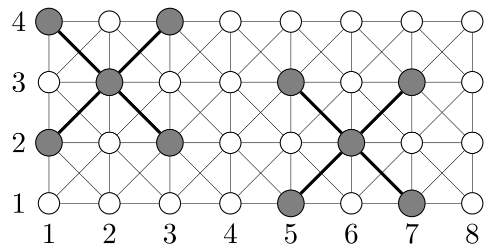

Figure 6: Tile generating an optimal identifying code of $\mathcal{K}_4$ (density 5/16)

Proof. The periodic identifying code of $\mathcal{K}_4$ generated by the tiles depicted in Figure 6 has density 5/16. Thus $d^*(\mathcal{K}_4) \leq 5/16$.

Let us now prove $d^*(\mathcal{K}_4) \geq 5/16$. Let C be an identifying code of $\mathcal{K}_4$. For every $a \in \mathbb{Z}$, let the initial charge $\mathrm{chrg}_0(a) = |Q_a \cap C|$. We say that an integer a is *satisfied* if its charge is at least 5/4, and *unsatisfied* otherwise. We apply three discharging rules, Rule i for $i = 1$ to 3 one after another. We denote by $\mathrm{chrg}_i(a)$ the charge of a after applying Rule i, and we set $\mathrm{exc}_i(a) = \mathrm{chrg}_i(a) - 5/4$ (observe that an integer is satisfied if and only if $\mathrm{exc}(a) \geq 0$).

> **[Rule** $1 \leq i \leq 3$]: every unsatisfied $a \in \mathbb{Z}$ receives $\min\{\mathrm{exc}_{i-1}(a-i), -\mathrm{exc}_{i-1}(a)\}$ from $a-i$, if $a-i$ is satisfied.

We shall prove that after applying the discharging rules, every $a \in \mathbb{Z}$ is satisfied. This yields $d(C, \mathcal{K}_4) \geq 5/16$. Observe that once an integer becomes satisfied, then its remains satisfied.

Assume for a contradiction that an integer a is unsatisfied after applying the three discharging rules. It was initially unsatisfied so $|Q_a \cap C| \leq 1$. By symmetry, we may assume that we are in one of the following cases: $Q_a \cap C = \emptyset$ (Case 1), $Q_a \cap C = \{(a, 1)\}$ (Case 2), $Q_a \cap C = \{(a, 2)\}$ (Case 3).

<u>**Case 1:**</u> $Q_a \cap C = \emptyset$. In this case, $\mathrm{exc}_0(a) = -5/4$. First, note that $(a-3, 1) \in C$ or $(a-3, 2) \in C$, since $C[(a-2, 1)] \neq C[(a-1, 1)]$, and $(a-3, 3) \in C$ or $(a-3, 4) \in C$, since $C[(a-2, 4)] \neq C[(a-1, 4)]$. In particular, $\mathrm{exc}_0(a-3) \geq 3/4$. Now $|Q_{a-1} \cap C| \leq 2$, for otherwise a receives 5/4 from $a-1$ by Rule 1, and is satisfied. Therefore, by symmetry, we are in one of the following seven subcases: $Q_{a-1} \cap C = \emptyset$ (Subcase 1.1), $Q_{a-1} \cap C = \{(a-1, 1)\}$ (Subcase 1.2), $Q_{a-1} \cap C = \{(a-1, 2)\}$ (Subcase 1.3), $Q_{a-1} \cap C = \{(a-1, 1), (a-1, 2)\}$ (Subcase 1.4), $Q_{a-1} \cap C = \{(a-1, 1), (a-1, 3)\}$ (Subcase 1.5), $Q_{a-1} \cap C = \{(a-1, 1), (a-1, 4)\}$ (Subcase 1.6), $Q_{a-1} \cap C = \{(a-1, 2), (a-1, 3)\}$ (Subcase 1.7).

<u>Subcase 1.1:</u> $Q_{a-1} \cap C = \emptyset$. Since $C[(a-1, 1)] \neq C[(a-1, 2)]$ and $C[(a-1, 4)] \neq C[(a-1, 3)]$, then $(a-2, 2), (a-2, 3) \in C$. Since $C[(a-1, 2)] \neq C[(a-1, 3)]$, then $(a-2, 1) \in C$ or $(a-2, 4) \in C$. In addition $|Q_{a-2} \cap C| \neq 4$, for otherwise $\mathrm{exc}_0(a-2) = 11/4$, so $a-2$ sends 5/4 to $a-1$ by Rule 1 and 5/4 to a by Rule 2, and a is satisfied. By symmetry, we may assume $(a-2, 1) \in C$ and $(a-2, 4) \notin C$. Thus $\mathrm{exc}_0(a-2) = 7/4$ and so $a-2$ sends 5/4 to $a-1$ by Rule 1 and sends 1/2 to a by Rule 2. Moreover, $a-3$ sends charge 1/2 to a by Rule 3, and so a is satisfied, a contradiction.

<u>Subcase 1.2:</u> $Q_{a-1} \cap C = \{(a-1, 1)\}$. Since $C[(a-1, 1)] \neq C[(a-1, 2)]$ and $C[(a-1, 4)] \neq C[(a-1, 3)]$, then $(a-2, 2), (a-2, 3) \in C$. In addition, $|Q_{a-2} \cap C| \leq 2$ for otherwise

$\mathrm{exc}_0(a-2) \geq \frac{7}{4}$, so $a-2$ sends $1/4$ to $a-1$ by Rule 1, and $5/4$ to a by Rule 2 and a is satisfied. Therefore $(a-2,1),(a-2,4) \notin C$. Consequently $a-2$ sends $1/4$ to $a-1$ (by Rule 1) satisfying it and sends $1/2$ to a by Rule 2. Finally, $a-3$ sends charge $3/4$ to a by Rule 3, and so a is satisfied, a contradiction.

<u>Subcase 1.3:</u> $Q_{a-1} \cap C = \{(a-1,2)\}$. Since $C[(a-1,1)] \neq C[(a-1,2)]$, then $(a-2,3) \in C$. Since $C[(a-1,2)] \neq C[(a-1,3)]$, then $(a-2,1) \in C$ or $(a-2,4) \in C$. If $(a-2,2) \in C$, then $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $5/4$ to a by Rule 2, so a is satisfied, a contradiction. Thus $(a-2,2) \notin C$. Now $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/2$ to a by Rule 2. Moreover, $a-3$ sends charge $3/4$ to a by Rule 3, so a is satisfied, a contradiction.

<u>Subcase 1.4:</u> $Q_{a-1} \cap C = \{(a-1,1),(a-1,2)\}$. Observe that $a-1$ sends $3/4$ to a by Rule 1, so $exc_1(a) = -1/2$. Since $C[(a-1,1)] \neq C[(a-1,2)]$, we have $(a-2,3) \in C$. But $|C \cap Q_{a-2}| \leq 1$, for otherwise $a-2$ sends $1/2$ to a by Rule 2, and a is satisfied. Thus $(a-2,1),(a-2,2),(a-2,4) \notin C$. Now $a-3$ sends $1/4$ to $a-2$ by Rule 1, and $1/2$ to a by Rule 3, so a is satisfied, a contradiction.

<u>Subcase 1.5:</u> $Q_{a-1} \cap C = \{(a-1,1),(a-1,3)\}$. Then $a-1$ sends $3/4$ to a by Rule 1, so $exc_1(a) = -1/2$. Since $C[(a-1,3)] \neq C[(a-1,4)]$, then $(a-2,2) \in C$. But $|C \cap Q_{a-2}| \leq 1$, for otherwise $a-2$ sends $1/2$ to a by Rule 2, and a is satisfied. Hence $(a-2,1),(a-2,3),(a-2,4) \notin C$. Now $a-3$ sends charge $1/4$ to $a-2$ by Rule 1, and $1/2$ to a by Rule 3, so a is satisfied, a contradiction.

<u>Subcase 1.6:</u> $Q_{a-1} \cap C = \{(a-1,1),(a-1,4)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$ implies $(a-2,3) \in C$. Moreover $C[(a-1,3)] \neq C[(a-1,4)]$, so $(a-2,2) \in C$. Thus, $a-1$ sends $3/4$ to a by Rule 1, and $a-2$ sends $1/2$ to a by Rule 2. Hence a is satisfied, a contradiction.

<u>Subcase 1.7:</u> $Q_{a-1} \cap C = \{(a-1,2),(a-1,3)\}$. Then $a-1$ sends $3/4$ to a by Rule 1, so $exc_1(a) = -1/2$. Moreover, $C[(a-1,2)] \neq C[(a-1,3)]$, so either $(a-2,1) \in C$ or $(a-2,4) \in C$. By symmetry, we may assume $(a-2,4) \in C$. But $|C \cap Q_{a-2}| \leq 1$, for otherwise $a-2$ sends $1/2$ to a by Rule 2, and a is satisfied. Hence $(a-2,1),(a-2,2),(a-2,3) \notin C$. Now $a-3$ sends charge $1/4$ to $a-2$ by Rule 1, and $1/2$ to a by Rule 3, so a is satisfied, a contradiction.

<u>Case 2:</u> $Q_a \cap C = \{(a,1)\}$. First, note that either $(a-3,3) \in C$ or $(a-3,4) \in C$, since $C[(a-2,4)] \neq C[(a-1,4)]$. Moreover $|Q_{a-1} \cap C| \leq 1$, for otherwise $a-1$ sends $1/4$ to a by Rule 1 and a is satisfied. Hence we are in one of the five subcases: $Q_{a-1} \cap C = \emptyset$ (Subcase 2.1), $Q_{a-1} \cap C = \{(a-1,1)\}$ (Subcase 2.2), $Q_{a-1} \cap C = \{(a-1,2)\}$ (Subcase 2.3), $Q_{a-1} \cap C = \{(a-1,3)\}$ (Subcase 2.4), $Q_{a-1} \cap C = \{(a-1,4)\}$ (Subcase 2.5).

<u>Subcase 2.1:</u> $Q_{a-1} \cap C = \emptyset$. Then $C[(a-1,1)] \neq C[(a-1,2)]$ and $C[(a-1,4)] \neq C[(a-1,3)]$, so $(a-2,2),(a-2,3) \in C$. But $|C \cap Q_{a-2}| \leq 2$, for otherwise $a-2$ sends $5/4$ to $a-1$ by Rule 1, and $1/4$ to a by Rule 2, and a is satisfied. Thus $(a-2,1),(a-2,4) \notin C$. In particular, $a-2$ sends $3/4$ to $a-1$ by Rule 1, so $\mathrm{exc}_1(a-1) = -1/2$. Moreover, $|C \cap Q_{a-3}| \leq 1$, for otherwise $a-3$ sends $1/2$ to $a-1$ by Rule 2 and $1/4$ to a by Rule 3 and a is satisfied. In particular, $(a-3,1),(a-3,2) \notin C$. Now $C[(a-2,2)] \neq C[(a-1,3)] = \{(a-2,2),(a-2,3)\}$ so $(a-3,3) \in C$, and $C[(a-2,3)] \neq C[(a-2,2)] = \{(a-2,2),(a-2,3),(a-3,3)\}$, so $(a-3,4) \in C$. This contradicts $|C \cap Q_{a-3}| \leq 1$.

<u>Subcase 2.2:</u> $Q_{a-1} \cap C = \{(a-1,1)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. Moreover $C[(a-1,3)] \neq C[(a-1,4)]$, so $(a-2,2) \in C$. Thus $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2. Hence a is satisfied, a contradiction.

Subcase 2.3: $Q_{a-1} \cap C = \{(a-1,2)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. But $|C \cap Q_{a-2}| \leq 1$, for otherwise $a-2$ sends $1/4$ to $a-1$ by Rule 1, and $1/4$ to a by Rule 2, and a is satisfied. Thus $(a-2,1),(a-2,2),(a-2,4) \notin C$. Moreover $|C \cap Q_{a-3}| \leq 1$ for otherwise $a-3$ sends $1/4$ to $a-2$, $a-1$, and a by Rules 1, 2 and 3, respectively, and a is satisfied. In particular, $(a-3,1),(a-3,2) \notin C$. Now $(a-3,3) \in C$ because $C[(a-2,2)] \neq C[(a-1,3)] = \{(a-1,2),(a-2,3)\}$, and $(a-3,4) \in C$ because $C[(a-2,3)] \neq C[(a-2,2)] = \{(a-1,2),(a-2,3),(a-3,3)\}$. This contradicts $|C \cap Q_{a-3}| \leq 1$.

Subcase 2.4: $Q_{a-1} \cap C = \{(a-1,3)\}$. Then $C[(a-1,4)] \neq C[(a-1,3)]$, so $(a-2,2) \in C$. But $|C \cap Q_{a-2}| \leq 1$ for otherwise $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2, and a is satisfied. Thus, $(a-2,1),(a-2,3),(a-2,4) \notin C$. Moreover $|C \cap Q_{a-3}| \leq 1$ for otherwise $a-3$ sends $1/4$ to $a-2$, $a-1$, and a by Rules 1, 2 and 3, respectively, and a is satisfied. In particular, $(a-3,1),(a-3,2) \notin C$. Now $(a-3,3) \in C$ because $C[(a-2,2)] \neq C[(a-1,3)] = \{(a-1,3),(a-2,2)\}$, and $(a-3,4) \in C$ because $C[(a-2,3)] \neq C[(a-2,2)] = \{(a-1,3),(a-2,2),(a-3,3)\}$. This contradicts $|C \cap Q_{a-3}| \leq 1$.

Subcase 2.5: $Q_{a-1} \cap C = \{(a-1,4)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. Moreover $C[(a-1,3)] \neq C[(a-1,4)]$, so $(a-2,2) \in C$. Hence $\mathrm{exc}_0(a-2) \geq 3/4$, thus $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2. Hence a is satisfied, a contradiction.

Case 3: $Q_a \cap C = \{(a,2)\}$. First, note that $(a-3,3) \in C$ or $(a-3,4) \in C$ because $C[(a-2,4)] \neq C[(a-1,4)]$. Moreover $|Q_{a-1} \cap C| \leq 1$, for otherwise $a-1$ sends $1/4$ to a by Rule 1 and a is satisfied. Hence we are in one of the five subcases: $Q_{a-1} \cap C = \emptyset$ (Subcase 3.1), $Q_{a-1} \cap C = \{(a-1,1)\}$ (Subcase 3.2), $Q_{a-1} \cap C = \{(a-1,2)\}$ (Subcase 3.3), $Q_{a-1} \cap C = \{(a-1,3)\}$ (Subcase 3.4), $Q_{a-1} \cap C = \{(a-1,4)\}$ (Subcase 3.5).

Subcase 3.1: $Q_{a-1} \cap C = \emptyset$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. Moreover $C[(a-1,2)] \neq C[(a-1,3)]$, so either $(a-2,1) \in C$ or $(a-2,4) \in C$. But $|C \cap Q_{a-2}| \leq 2$ for otherwise $a-2$ sends $5/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2, and a is satisfied. In particular, $|C \cap Q_{a-2}| = 2$, $a-2$ sends $3/4$ to $a-1$ by Rule 1, and so $\mathrm{exc}_1(a-1) = -1/2$. Moreover $(a-2,2) \notin C$. Now $C[(a-2,3)] \neq C[(a-2,4)]$, so $(a-3,2) \in C$. Thus $|C \cap Q_{a-3}| \geq 2$. Consequently, $a-3$ sends $1/2$ to $a-1$ by Rule 2 and $1/4$ to a by Rule 3. Hence a is satisfied, a contradiction.

Subcase 3.2: $Q_{a-1} \cap C = \{(a-1,1)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. But $|C \cap Q_{a-2}| \leq 1$ for otherwise $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2, and a is satisfied. Thus $(a-2,1),(a-2,2),(a-2,4) \notin C$. Now $C[(a-2,3)] \neq C[(a-2,4)]$, so $(a-3,2) \in C$. Hence $|C \cap Q_{a-3}| \geq 2$. Thus $a-3$ sends $1/4$ to $a-2$, $a-1$ and a by Rules 1, 2 and 3, respectively. Hence a is satisfied, a contradiction.

Subcase 3.3: $Q_{a-1} \cap C = \{(a-1,2)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, then $(a-2,3) \in C$. Since $C[(a-1,2)] \neq C[(a-1,3)]$, then either $(a-2,1) \in C$ or $(a-2,4) \in C$. With this, $a-2$ satisfies $a-1$ and a sends charge $1/4$ to them in Rules 1 and 2, respectively, and we are done.

Subcase 3.4: $Q_{a-1} \cap C = \{(a-1,3)\}$. Then $C[(a-1,2)] \neq C[(a-1,3)]$, so either $(a-2,1) \in C$ or $(a-2,4) \in C$. But $|C \cap Q_{a-2}| \leq 1$ for otherwise $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2, and a is satisfied. Thus $(a-2,2),(a-2,3) \notin C$. Now $C[(a-2,3)] \neq C[(a-2,4)]$, so $(a-3,2) \in C$. Hence $|C \cap Q_{a-3}| \geq 2$. Thus $a-3$ sends $1/4$ to $a-2$, $a-1$ and a by Rules 1, 2 and 3, respectively. Hence a is satisfied, a contradiction.

Subcase 3.5: $Q_{a-1} \cap C = \{(a-1,4)\}$. Then $C[(a-1,1)] \neq C[(a-1,2)]$, so $(a-2,3) \in C$. But $|C \cap Q_{a-2}| \leq 1$ for otherwise $a-2$ sends $1/4$ to $a-1$ by Rule 1 and $1/4$ to a by Rule 2, and a is satisfied. Thus $(a-2,1),(a-2,2),(a-2,4) \notin C$. Now $C[(a-2,3)] \neq C[(a-2,4)]$, so $(a-3,2) \in C$. Hence $|C \cap Q_{a-3}| \geq 2$. Thus $a-3$ sends $1/4$ to $a-2$, $a-1$ and a by Rules 1, 2 and 3, respectively. Hence a is satisfied, a contradiction. $\square$

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